10 On the Existence of a Stationary Optimal Stock for a Multi-sector Economy with Nonconvex Technology*

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1 INTRODUCTION

In 1967, David Gale's seminal paper, 'On Optimal Development in a Multi-Sector Economy', presented a general result on the existence of optimal programmes over an infinite horizon, when future utilities are not discounted. This paper changed, in a very substantive manner, the way in which economists approached the problems of optimal allocation of resources over time. The question with which the paper concerned itself was, of course, a very challenging one: it had engaged the attention of economists like Ramsey, Samuelson and Solow, Koopmans and Weizsacker, among others, over roughly a forty-year period. But the enduring importance of the paper lies less in the specific 'answer' obtained, and more on the techniques used to obtain it. In effect, David Gale showed that the main propositions of optimal growth theory could be established by reasoning, both elegant and compelling, with a few simple results from the theory of convex analysis.

The basic technique used in Gale's paper is the theory of *dual variables*, or 'the method of *price-systems*', (to borrow an expression from Gale, 1968). Application of this technique to an optimal growth model with a convex technology set and a concave utility function helped to bring to centre stage a single remarkably useful construct: a golden rule supported by a suitable price system. The profound influence of Gale's methods in general, and of this construct in particular, on subsequent research in this area is clear from a survey by McKenzie (1986).

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The present study can be viewed as an attempt to extend the analysis of Gale to multi-sectoral economies which allow for significant *nonconvexities* in the technology set. The pervasive use of David Gale's methods throughout the study will be obvious even to the casual reader. But the fact that the construct of a price-supported golden-rule continues to hold centre stage in a model with a nonconvex technology set might come as a surprise to some. Certainly, it provides another instance where a construct emphasised by Gale has been found to be significant in an environment different from the one which gave rise to it.

The literature on optimal intertemporal allocation of resources with a nonconvex technology has primarily confined itself to the framework of an aggregative model. The contributions have emphasised the qualitative differences between the theory with production nonconvexities, and the corresponding theory under convex production structures (see particularly Clark, 1971; Skiba, 1978; Majumdar and Mitra, 1982; 1983); they have introduced new techniques in dynamic optimisation theory to deal with the technical problems posed by nonconvex structures (see, for example, Dechert and Nishimura, 1983; Majumdar and Nermuth, 1982), which have also turned out to be useful in re-examining some problems under convex structures (see Mitra, 1983); a unification of the results for convex and nonconvex technology sets has also been obtained (see Mitra and Ray, 1984).

I believe that the aggregative model with a nonconvex technology set is sufficiently well understood by this time, so that one can venture ahead to study some problems that arise in multi-sector models under nonconvexities. This paper is a study of *one* such problem: the existence of a nontrivial stationary optimal stock, when future utilities are *not* discounted. (I should mention that Ekeland and Scheinkman's, 1983, paper is the only one I know of which deals with dynamic optimisation is a nonconvex multi-sectoral setting; however, they are concerned with the existence of optimal programmes, and the necessity of some version of the 'transversality condition' being satisfied by optimal programmes, when future utilities are discounted.)

The existence of a nontrivial stationary optimal stock can be established in *convex* models by first establishing the existence of a golden rule – that is, a stationary programme which maximises period utility among all stationary programmes. A golden rule can then be supported by a price, using a standard separation theorem for convex sets. Finally, programmes which are 'good' (that is, which do not become 'infinitely worse' than a golden rule programme in the sum of utilities as the horizon approaches infinity) can be shown to satisfy an 'average turnpike property' (that is, the long-run average input and output levels converge to those of the set of golden rules). These three steps can be combined to show that a golden rule programme is a stationary optimal programme in the sense of the 'overtaking criterion'. For the sake of keeping our exposition self-contained, we establish the existence of a golden rule (this does *not* involve convex structures) in section 3, and the existence of a price-support for a golden rule and a stationary optimal programme for *convex* models in section 4. (The methods used are of course familiar through the well-known papers of Gale, 1967; McKenzie, 1968; Brock, 1970; and Peleg 1973).

The problem that arises when we allow the technology set to be non-convex is illustrated by a two-good example in section 5. In the example, there is no stationary optimal programme and, in fact, it is shown that given any nontrivial stationary programme an oscillating programme can be found which is better than the stationary one. On the other hand, the existence of a nontrivial stationary optimal programme can be proved without the assumption of a convex technology set in the aggregative model. We demonstrate this in section 6, since it is a result not readily available in the literature in as general a form as we are asserting here.

Our basic observation in section 7 is that in the existence proof for convex models, convexity of the technology set is *not* essential to proving a price-support for some golden rule. It can be accomplished as well when the technology set is 'quasi-star-shaped' with respect to some golden rule (besides satisfying a couple of regularity assumptions, namely strictly positive consumption at every golden rule, and smoothness of the utility function). We present examples to show that each of these assumptions is indispensable in providing a price-support for a golden rule.

It apppears to us that the 'average turnpike property' proved in convex models is critically dependent on the convexity of the technology set. We, therefore, avoid this route and follow in section 8 an alternative route, using the so-called 'value-loss lemma'. This requires us to strengthen our assumption and make the technology set 'strictly quasi-star-shaped' with respect to some golden rule. The existence result then follows by applying fairly standard methods used in the literature. While the assumption of a 'strictly quasi-star-shaped' technology (with respect to some golden rule) is surely a restriction, we verify in section 8 that in the canonical ('S-shaped' production function) case studied in the aggregative model by Clark (1971), Skiba (1978), Majumdar and Mitra (1982; 1983), Dechert and Nishimura (1983) and others, this assumption is automatically satisfied.

2 THE MODEL

The model is described by a pair (Ω, w) where Ω is a subset of $\mathscr{R}_+^n \times \mathscr{R}_+^n$, and w is a function from $\mathscr{R}_+^n$ to $\mathscr{R}$. Ω is interpreted as a *technology set*, and w as a *utility* function.

A *programme* from k in $\mathscr{R}_+^n$ is a sequence $\{x(t), y(t+1)\}_0^\infty$ such that

$$x(0)= k;\ (x(t),\ y(t+1)) \text{ is in } \Omega \text{ for } t \geqslant 0;\ y(t) \geqslant x(t) \text{ for } t \geqslant 1$$

Associated with a programme $\{x(t),\ y(t+1)\}_0^\infty$ from k, is a sequence $\{c(t)\}_0^\infty$ defined by

$$c(t) = y(t) - x(t) \text{ for } t \geqslant 1$$

We interpret (x, y, c) as the *input*, *output* and *consumption* levels respectively.

A programme $\{x(t),\ y(t+1)\}_0^\infty$ from k is *stationary* if

$$x(t) = k \text{ for } t \geqslant 0,\ y(t) = y(t+1) \text{ for } t \geqslant 1$$

It follows that the associated (consumption) sequence $\{c(t)\}_0^\infty$ satisfies

$$c(t) = c(t+1) \text{ for } t \geqslant 1$$

for a stationary programme.

A programme $\{x^*(t),\ y^*(t+1)\}_0^\infty$ from k is *optimal* if

$$\liminf_{T\to\infty} \sum_{t=1}^{T} [w(c(t)) - w(c^*(t))] \leqslant 0$$

for every programme $\{x(t),\ y(t+1)\}_0^\infty$ from k.

A programme $\{x(t), y(t+1)\}_0^\infty$ from k is a *stationary optimal programme* (SOP) if it is both stationary and optimal. It is a *nontrivial stationary optimal programme* if it is a SOP and $w(c(1)) > w(0)$.

A *stationary optimal stock* (SOS) is an element k in $\mathscr{R}_+^n$, such that there is a stationary optimal programme from k. It is a *nontrivial stationary optimal stock* if there is a nontrivial stationary optimal programme from k.

The following assumptions are made on Ω and w:

Assumption 1 $(0, 0)$ is in Ω; $(0, y)$ in Ω implies $y = 0$.

Assumption 2 If (x, y) is in Ω, and $x' \geqslant x$, and $0 \leqslant y' \leqslant y$, then (x', y') is in Ω.

Assumption 3 Ω is closed.

Assumption 4 There is $\beta > 0$, such that (x, y) in Ω and $\|x\| > \beta$ implies $\|y\| < \|x\|$.

Assumption 5 There is $(\bar{x}, \bar{y})$ in Ω, with $\bar{y} >> \bar{x}$.

Assumption 6 w is continuous and concave on $\mathscr{R}_+^n$.

Assumption 7 If $c' \geqslant c \geqslant 0$, then $w(c') \geqslant w(c)$; if $c' >> c$, then $w(c') > w(c)$.

We now prove a convenient boundedness property which will be used subsequently.

Lemma 2.1 *If* (x, y) is in Ω, and $\|x\| \leqslant \beta$, then $\|y\| \leqslant \beta$.

Proof Suppose, on the contrary, there is (x, y) in Ω, with $\|x\| \leqslant \beta$, but $\|y\| > \beta$. Then, we have $0 < \|x\| \leqslant \beta$; so, defining $x^s = [\beta + (1/s)][x/\|x\|]$, we have $x^s \geqslant x$, and $\|x^s\| = \beta + (1/s) > \beta$, for $s = 1, 2, \ldots$. Since (x, y) is in Ω, (x^s, y) is in Ω for each s, and $\|y\| < \|x^s\|$ by Assumption 4. Since $x^s \to \beta x/\|x\| \equiv \bar{x}$ as $s \to \infty$, so $\|y\| \leqslant \|\bar{x}\| = \beta$, a contradiction.

Corollary 2.1 *If* (x, y) is in Ω, then $\|y\| \leqslant \max(\|x\|, \beta)$.

Proof This follows from Assumption 4 and Lemma 2.1.

Corollary 2.2 If $\{x(t), y(t + 1)\}_0^\infty$ is a programme from k, and $\|k\| \leqslant \beta$, then $\|x(t)\| \leqslant \beta$ for $t \geqslant 0$.

Proof Clearly $\|x(0)\| = \|k\| \leqslant \beta$. Also, if $\|x(t)\| \leqslant \beta$ for some $t \geqslant 0$, then by Lemma 2.1, $\|y(t + 1)\| \leqslant \beta$. Since $0 \leqslant x(t + 1) \leqslant y(t + 1)$, we obtain $\|x(t + 1)\| \leqslant \beta$. This completes the proof by induction.

Remark It is clear from Corollary 2.2, that if $\{x(t), y(t + 1)\}_0^\infty$ is a programme from k, and $\|k\| \leqslant \beta$, then $\|y(t)\| \leqslant \beta$, and $\|c(t)\| \leqslant \beta$ for $t \geqslant 1$.

Corollary 2.3 If $\{x(t), y(t + 1)\}_0^\infty$ is a programme from k, then $\|x(t)\| \leqslant \max[\|k\|, \beta]$ for $t \geqslant 0$.

Proof Define $\beta' \equiv \max[\|k\|, \beta]$. Clearly, $\|x(0)\| = \|k\| \leqslant \beta'$. Also, if $\|x(t)\| \leqslant \beta'$ for some $t \geqslant 0$, then by Corollary 2.1, $\|y(t + 1)\| \leqslant \max(\|x(t)\|, \beta) \leqslant \max(\beta', \beta) = \beta'$. Since $0 \leqslant x(t + 1) \leqslant y(t + 1)$, we obtain $\|x(t + 1)\| \leqslant \beta'$. This completes the proof by induction.

Remark It is clear that if $\{x(t), y(t + 1)\}_0^\infty$ is a programme from k, then $\|y(t)\| \leqslant \max[\|k\|, \beta]$, and $\|c(t)\| \leqslant \max[\|k\|, \beta]$ for $t \geqslant 1$.

3 A GOLDEN RULE

A pair $(\hat{x}, \hat{y})$ in Ω is called a *golden rule* if $\hat{y} \geqslant \hat{x}$ and $w(\hat{y} - \hat{x}) \geqslant w(y - x)$ for all (x, y) in Ω satisfying $y \geqslant x$.

Let $A = \{(x, y) \text{ in } \Omega: y \geqslant x\}$. Then A is nonempty, since $(0, 0)$ is in Ω. If (x, y) is in Ω with $\|x\| > \beta$, then $\|y\| < \|x\|$; so, for all (x, y) in A, $\|x\| \leqslant \beta$. By Lemma 2.1, for all (x, y) in A, $\|y\| \leqslant \beta$. Thus A is bounded. A is closed since Ω is closed.

Let $B = \{c: c = y - x$, for (x, y) in $A\}$. Since A is nonempty and compact, B is also nonempty and compact. Since B is a subset of $\mathscr{R}_+^n$, w is continuous on B and there is $\hat{c}$ in B such that $w(\hat{c}) \geq w(c)$ for all c in B. That is, there is $(\hat{x}, \hat{y})$ in Ω, with $\hat{y} \geq \hat{x}$, such that $w(\hat{y} - \hat{x}) \geq w(y - x)$ for all (x, y) in Ω, with $y \geq x$. Hence $(\hat{x}, \hat{y})$ is a golden rule. By definition $w(\hat{y} - \hat{x}) \geq w(\bar{y} - \bar{x}) > w(0)$, since $\bar{y} >> \bar{x}$ by Assumption 5, and w is increasing in the sense of Assumption 7.

Denote the utility level of a golden rule by $\hat{w}$. Following Gale (1967), a programme $\{x(t), y(t + 1)\}_0^\infty$ from k in $\mathscr{R}_+^n$ is called *good* if there is a real number, Θ, such that

$$\sum_{t=1}^{T} [w(c(t)) - \hat{w}] \geq \Theta \quad \text{for } T \geq 1$$

If $(\hat{x}, \hat{y})$ is a golden rule, and there is $\hat{p}$ in $\mathscr{R}_+^n$ such that

(i) $w(\hat{y} - \hat{x}) - \hat{p}(\hat{y} - \hat{x}) \geq w(c) - \hat{p}c$ for all $c \in \mathscr{R}_+^n$
(ii) $\hat{p}\hat{y} - \hat{p}\hat{x} \geq \hat{p}y - \hat{p}x$ for all $(x, y) \in \Omega$

we say that the golden rule is *price-supported* and $\hat{p}$ is called a *price-support* of the golden rule $(\hat{x}, \hat{y})$.

4 THE EXISTENCE OF A PRICE-SUPPORTED GOLDEN RULE AND A STATIONARY OPTIMAL STOCK FOR A CONVEX TECHNOLOGY

As mentioned in section 1, a basic contribution of Gale (1967) was to emphasise the central role of the theory of *dual variables* for the existence of optimal programmes in convex models. The application of duality theory brought into sharp focus the importance of a *price-supported golden rule*; while the other applications of duality theory in Gale's paper were of interest on their own, Brock (1970) showed that they were not indispensable to proving the existence of an optimal programme. Brock used the assumption of the *uniqueness* of the golden rule to establish his existence result, and subsequently Peleg (1973) showed that this assumption can also be dispensed with in proving the existence of a *stationary* optimal programme. All these papers work in a framework which McKenzie (1986) calls the 'reduced' form. We provide here a self-contained theory of the existence of a price-supported golden rule and a nontrivial stationary optimal stock for the case where the technology set, Ω, is convex, and where utility is derived from consumption alone. Of course, we rely heavily on the techniques used in the above-mentioned papers.

Lemma 4.1 Let $(\hat{x}, \hat{y})$ be a golden rule. If Ω is convex, then there is $\hat{p} > 0$ in $\mathscr{R}_+^n$, such that

(i) $w(\hat{y} - \hat{x}) - \hat{p}(\hat{y} - \hat{x}) \geq w(c) - \hat{p}c$ for all c in $\mathscr{R}^n_+$
(ii) $\hat{p}\hat{y} - \hat{p}\hat{x} \geq \hat{p}y - \hat{p}x$ for all (x, y) in Ω

Proof Let $\hat{A} = \{c$ in $\mathscr{R}^n$: $c = y - x$, where (x, y) is in $\Omega\}$, $\hat{B} = \{c$ in $\mathscr{R}^n_+$: $w(c) > w(\hat{y} - \hat{x})\}$. $\hat{A}$ is convex since Ω is convex; $\hat{B}$ is convex since w is concave. $\hat{A}$ and $\hat{B}$ are nonempty. Furthermore $\hat{A}$ and $\hat{B}$ are disjoint, by the definition of a golden rule. Hence, there is $\hat{q}$ in $\mathscr{R}^n$, $\hat{q} \neq 0$, and Θ in $\mathscr{R}$ such that

$$\hat{q}c \leq \Theta \text{ for all } c \text{ in } \hat{A} \tag{10.4.1}$$

$$\hat{q}c \geq \Theta \text{ for all } c \text{ in } \hat{B} \tag{10.4.2}$$

Using (10.4.1), and the fact that $(\hat{y}, \hat{x})$ is in $\hat{A}$, we have $\hat{q}(\hat{y} - \hat{x}) \leq \Theta$. Using (10.4.2), and the fact that $(\hat{y} - \hat{x}) + (1/s)e$ is in B for $s = 1, 2, 3, \ldots$ (where $e = (1, 1, 1, \ldots, 1)$ in $\mathscr{R}^n$), we have $\hat{q}(\hat{y} - \hat{x}) \geq \Theta$. Hence $\Theta = \hat{q}(\hat{y} - \hat{x})$. By (10.4.2), $\hat{q}$ is in $\mathscr{R}^n_+$, and since $\hat{q} \neq 0$ we have $\hat{q} > 0$. By (10.4.1) and (10.4.2), we have

$$\hat{q}(y - x) \leq \hat{q}(\hat{y} - \hat{x}) \text{ for all } (x, y) \text{ in } \Omega \tag{10.4.3}$$

$$\hat{q}c \geq \hat{q}(\hat{y} - \hat{x}) \text{ for all } c \text{ in } \mathscr{R}^n_+, \text{ with } w(c) > w(\hat{y} - \hat{x}) \tag{10.4.4}$$

Note that since $(\bar{x}, \bar{y})$ is in Ω, with $\bar{y} >> \bar{x}$, we get

$$\hat{q}(\hat{y} - \hat{x}) > 0 \tag{10.4.5}$$

using (10.4.3) and $\hat{q} > 0$.

Define, next, the sets $A' = \{(a, b)$ in $\mathscr{R}^2$: $a \leq w(c) - w(\hat{y} - \hat{x})$, $b \leq \hat{q}(\hat{y} - \hat{x}) - \hat{q}c$, for some c in $\mathscr{R}^n_+\}$ and $B' = \{(a, b)$ in $\mathscr{R}^2$: $(a, b) >> 0\}$. Clearly A is nonempty and convex. By (10.4.4), A' and B' are disjoint. Hence, there is (P, Q) in $\mathscr{R}^2_+$, $(P, Q) \neq 0$ such that

$$Pa + Qb \leq 0 \text{ for all } (a, b) \text{ in } A' \tag{10.4.6}$$

We claim that $P \neq 0$. For, if $P = 0$, then $Q > 0$, and by (10.4.6), $Qb \leq 0$ for all (a, b) in A', so that

$$b \leq 0 \text{ for all } (a, b) \text{ in } A' \tag{10.4.7}$$

Let $c = \frac{1}{2}(\hat{y} - \hat{x})$, $b = \frac{1}{2}\hat{q}(\hat{y} - \hat{x})$, and $a = w(\frac{1}{2}(\hat{y} - \hat{x})) - w(\hat{y} - \hat{x})$. Then (a, b) is in A', and by (10.4.5), $b > 0$, which contradicts (10.4.7). Hence $P \neq 0$; since $P \geq 0$, we have $P > 0$. Using (10.4.6), then, we have

$$a + (Q/P)b \leqslant 0 \text{ for all } (a, b) \text{ in } A' \tag{10.4.8}$$

Defining $\hat{p} = (Q/P)\hat{q}$, we have $\hat{p}$ in $\mathscr{R}_+^n$ and

$$[w(c) - w(\hat{y} - \hat{x})] + \hat{p}[(\hat{y} - \hat{x}) - c] \leqslant 0 \text{ for all } c \text{ in } \mathscr{R}_+^n \tag{10.4.9}$$

Rewriting this, we have

$$w(\hat{y} - \hat{x}) - \hat{p}(\hat{y} - \hat{x}) \geqslant w(c) - \hat{p}c \text{ for all } c \text{ in } \mathscr{R}_+^n \tag{10.4.10}$$

Using Assumption 7 in (10.4.10), we get $\hat{p} > 0$. Furthermore, using (10.4.3) and the definition of $\hat{p}$,

$$\hat{p}\hat{y} - \hat{p}\hat{x} \geqslant \hat{p}y - \hat{p}x \quad \text{for all } (x, y) \text{ in } \Omega \tag{10.4.11}$$

This completes the proof of the lemma.

Following the method of Brock (1970), one can prove the 'average turnpike property' of good programmes.

Lemma 4.2. Suppose Ω is convex. If $\{x(t), y(t+1)\}_0^\infty$ is a good programme from k in $\mathscr{R}_+^n$, and

$$\left.\begin{aligned} \bar{x}(T) &= [x(1) + \ldots + x(T)]/T \\ \bar{y}(T) &= [y(1) + \ldots + y(T)]/T \end{aligned}\right\} \quad \text{for } T \geqslant 1$$

and $(\bar{x}, \bar{y})$ is any accumulation point of $\{\bar{x}(T), \bar{y}(T)\}_1^\infty$, then $(\bar{x}, \bar{y})$ is a golden rule.

Proof Let $(\bar{x}, \bar{y})$ be an accumulation point of $\{\bar{x}(T), \bar{y}(T)\}_1^\infty$. Since $y(t) \geqslant x(t)$ for $t \geqslant 1$, we get $\bar{y}(T) \geqslant \bar{x}(T)$ for $T \geqslant 1$, and $\bar{y} \geqslant \bar{x}$.

Also, $(x(t), y(t+1))$ is in Ω for $t \geqslant 0$, so $([x(0) + \ldots + x(T-1)]/T, [y(1) + \ldots + y(T)]/T)$ is in Ω for $T \geqslant 1$, by convexity of Ω. Now, for $T \geqslant 1$,

$$\bar{x}(T) = ([x(0) + \ldots + x(T-1)]/T) + ([x(T) - x(0)]/T)$$

$$\bar{y}(T) = [y(1) + \ldots + y(T)]/T$$

Thus, if $(\bar{x}, \bar{y})$ is an accumulation point of $\{\bar{x}(T), \bar{y}(T)\}_1^\infty$, it is also an accumulation point of $\{[x(0) + \ldots + x(T-1)]/T, [y(1) + \ldots + y(T)]/T\}_1^\infty$. Since Ω is closed, $(\bar{x}, \bar{y})$ is in Ω.

Since $\{x(t), y(t+1)\}_1^\infty$ is good and w is concave, there is a real number Θ, such that for $T \geqslant 1$,

$$w(\bar{y}(T) - \bar{x}(T)) - \hat{w} \geqslant \sum_{t=1}^{T} [(w(y(t) - x(t)) - \hat{w})/T] \geqslant (\Theta/T) \quad (10.4.12)$$

Hence $w(\bar{y} - \bar{x}) - \hat{w} \geqslant 0$. Since $(\bar{x}, \bar{y})$ is in Ω, and $\bar{y} - \bar{x} \geqslant 0$, we have $w(\bar{y} - \bar{x}) - \hat{w} \leqslant 0$, by definition of a golden rule. Hence $w(\bar{y} - \bar{x}) = \hat{w}$, and so $(\bar{x}, \bar{y})$ is a golden rule.

Lemma 4.3 Suppose $(\hat{x}, \hat{y})$ in Ω is a golden rule, with price-support $\hat{p}$. Suppose, further, that $(\tilde{x}, \tilde{y})$ in Ω is also a golden rule. Then $\hat{p}$ is a price-support for $(\tilde{x}, \tilde{y})$.

Proof Since $(\hat{x}, \hat{y})$ and $(\tilde{x}, \tilde{y})$ are golden rules, we have $w(\hat{y}, \hat{x}) = w(\tilde{y} - \tilde{x})$. Since $\hat{p}$ is a price-support for $(\hat{x}, \hat{y})$, we get by (i) of Lemma 4.1, $\hat{p}(\tilde{y} - \tilde{x}) \geqslant \hat{p}(\hat{y} - \hat{x})$. By (ii) of Lemma 4.1, we have $\hat{p}(\hat{y} - \hat{x}) \geqslant \hat{p}(\tilde{y} - \tilde{x})$. So $\hat{p}(\hat{y} - \hat{x}) = \hat{p}(\tilde{y} - \tilde{x})$. Hence, using (i) of Lemma 4.1, we have

$$w(\tilde{y} - \tilde{x}) - \hat{p}(\tilde{y} - \tilde{x}) = w(\hat{y} - \hat{x}) - \hat{p}(\hat{y} - \hat{x}) \geqslant w(c) - \hat{p}c \quad \text{for all } c \text{ in } \mathscr{R}_+^n$$

Using (ii) of Lemma 4.1, we have

$$\hat{p}\tilde{y} - \hat{p}\tilde{x} = \hat{p}\hat{y} - \hat{p}\hat{x} \geqslant \hat{p}y - \hat{p}x \quad \text{for all } (x, y) \text{ in } \Omega$$

Hence, we have verified that $\hat{p}$ is a price-support for $(\tilde{x}, \tilde{y})$.

Let $(\hat{x}, \hat{y})$ be a golden rule with price-support $\hat{p}$. Let $A^0 = \{(x, y) \text{ in } \Omega\text{: } (x, y) \text{ is a golden rule}\}$. Let $B^0 = \{(x', y') \text{ in } A^0\text{: } \hat{p}x' \leqslant \hat{p}x \text{ for all } (x, y) \text{ in } A^0\}$. Clearly, A^0 is nonempty and compact. Hence, B^0 is nonempty. We will now show that any element (x', y') of B^0 is a golden rule with the property that x' is a nontrivial stationary optimal stock.

Theorem 4.1 Suppose the technology set Ω is convex. Then, there exists a golden rule (x', y'), such that x' is a nontrivial stationary optimal stock.

Proof Since Ω is convex, there is a golden rule $(\hat{x}, \hat{y})$ with price support $\hat{p}$. We may define B^0 as above, and verify that it is nonempty. Let (x', y') be any element of B^0. Then (x', y') is a golden rule, with price support $\hat{p}$, by Lemma 4.3. Also,

$$\hat{p}x' \leqslant \hat{p}x \quad (10.4.13)$$

where (x, y) is any golden rule.

Define $\{x'(t), y'(t + 1)\}_0^\infty$ by $x'(t) = x'$ and $y'(t + 1) = y'$ for $t \geqslant 0$.

Clearly, $\{x'(t), y'(t+1)\}_0^\infty$ is a stationary programme from x'. We now claim that this is an optimal programme from x'. Suppose not. Then, there is a programme $\{x(t), y(t+1)\}_0^\infty$ from x', a real number $\Theta' > 0$, and an integer $\bar{N} \geqslant 1$, such that

$$\Theta' \leqslant \sum_{t=1}^{N} [w(c(t)) - w(c'(t))] = \sum_{t=1}^{N} [w(c(t)) - \hat{w}] \text{ for } N \geqslant \bar{N} \quad (10.4.14)$$

Then $\{x(t), y(t+1)\}_0^\infty$ is a good programme.

Using the fact that $\hat{p}$ is a price-support of (x', y'), we have for $N \geqslant 1$,

$$\begin{aligned}\sum_{t=1}^{N} [w(c(t)) - \hat{w}] &\leqslant \sum_{t=1}^{N} \hat{p}[(y(t) - x(t)) - (y' - x')] \\ &= \sum_{t=1}^{N} ([\hat{p}y(t) - \hat{p}x(t-1)] - [\hat{p}y' - \hat{p}x']) \\ &+ \hat{p}x(0) - \hat{p}x(N) \\ &\leqslant \hat{p}x' - \hat{p}x(N)\end{aligned}$$

So, combining this with (10.4.14), we have

$$\hat{p}x' \geqslant \hat{p}x(N) + \Theta' \quad \text{for } N \geqslant \bar{N} \quad (10.4.15)$$

Let $(\bar{x}(T), \bar{y}(T)) = ([x(1) + \ldots . + x(T)]/T, [y(1) + \ldots . + y(T)]/T)$, for $T \geqslant 1$. Then, there is $\tilde{N} \geqslant \bar{N}$, such that

$$\hat{p}x' \geqslant \hat{p}\bar{x}(N) + (\Theta'/2) \quad \text{for } N \geqslant \tilde{N} \quad (10.4.16)$$

Since $\{x(t), y(t+1)\}_0^\infty$ is a good programme, any accumulation point $(\bar{x}, \bar{y})$ of $\{(\bar{x}(T), \bar{y}(T))\}_1^\infty$ is a golden rule by Lemma 4.2, and by (10.4.13),

$$\hat{p}x' \leqslant \hat{p}\bar{x} \quad (10.4.17)$$

On the other hand, by using (10.4.16), we have

$$\hat{p}x' \geqslant \hat{p}\bar{x} + (\Theta'/2) \quad (10.4.18)$$

which contradicts (10.4.17) and establishes that $\{(x'(t), y'(t+1)\}_0^\infty$ is an optimal programme. So x' is a stationary optimal stock. Since (x', y') is a golden rule, and there is $(\bar{x}, \bar{y})$ in Ω with $\bar{y} >> \bar{x}$, we have

$w(y' - x') \geqslant w(\bar{y} - \bar{x}) > w(0)$ by Assumption 7. Hence x' is a nontrivial stationary optimal stock.

5 THE NONEXISTENCE OF A NONTRIVIAL STATIONARY OPTIMAL STOCK FOR A NONCONVEX TECHNOLOGY

When the technology set is nonconvex, the existence theorem of section 4 breaks down. We provide in section 5 an example involving two goods, where the technology set, Ω, and the utility function, w, satisfy Assumptions 1–7, but there is no nontrivial stationary optimal stock.

Example 5.1

$$\text{Let } A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, B = \begin{bmatrix} 2 & 4 \\ 1 & 0 \end{bmatrix}, e = (1, 1)$$

Define the technology set $\Omega = \{(x, y) \in \mathscr{R}_+^2 \times \mathscr{R}_+^2 \colon Az \leqslant x, Bz \geqslant y, ez \leqslant 1$ and $\min(z_1, z_2) = 0$ for some $z \in \mathscr{R}_+^2\}$. Define the utility function $w(c_1, c_2) = c_1 + c_2$ for all $(c_1, c_2) \in \mathscr{R}_+^2$. Then (Ω, w) satisfy assumptions 1–7.

We note first that $\hat{x} = (1, 0)$, $\hat{y} = (2, 1)$, $\hat{c} = (1, 1)$, $\hat{z} = (1, 0)$ and $w(\hat{c}) = 2$ characterises the unique golden rule. To see this, observe that if $\hat{z}_2 > 0$, then $\hat{z}_1 = 0$, and $A\hat{z} = (0, \hat{z}_2)$, while $B\hat{z} = (4\hat{z}_2, 0)$. But, then $\hat{y} \geqslant \hat{x}$ is contradicted, since $\hat{y}_2 \leqslant 0 < \hat{z}_2 \leqslant \hat{x}_2$. Thus, $\hat{z}_2 = 0$ and $\hat{z}_1 \leqslant 1$. In this case, $\hat{y} \leqslant (2\hat{z}_1, \hat{z}_1)$ and $\hat{x} \geqslant (\hat{z}_1, 0)$, so $\hat{c} \leqslant (\hat{z}_1, \hat{z}_1)$ and $w(\hat{c}) \leqslant 2\hat{z}_1$. Since this is clearly a maximum when $\hat{z}_1 = 1$, we must have $\hat{z} = (1, 0)$. It follows readily now that $\hat{x} = (1, 0)$, $\hat{y} = (2, 1)$, $\hat{c} = (1, 1)$ and $w(\hat{c}) = 2$.

Second, we note that the golden rule does not have a price-support. For if $\hat{p}$ was a price-support of $(\hat{x}, \hat{y})$, then we must have $\hat{p} = (1, 1)$ since $w(c_1, c_2) = c_1 + c_2$ and $\hat{c} >> 0$. Thus $\hat{p}\hat{y} - \hat{p}\hat{x} = 3 - 1 = 2$. Now, let $z = (0, 1)$, $x = Az$, $y = Bz$. Then $(x, y) \in \Omega$, and further $x = (0, 1)$ while $y = (4, 0)$. Thus $\hat{p}y - \hat{p}x = 4 - 1 = 3 > 2 = \hat{p}\hat{y} - \hat{p}\hat{x}$, a contradiction.

Finally, we note that there is no nontrivial stationary optimal stock. For if $x > 0$ is such a stock, then there is $y \in \mathscr{R}_+^2$, $z \in \mathscr{R}_+^2$ such that $(x, y) \in \Omega$ and $y \geqslant x$, and by the argument used above, $z_2 = 0$ and $0 < z_1 \leqslant 1$. Thus, along the corresponding stationary programme there is a stationary utility level of $w(y - x) \leqslant 2z_1$. Now, we define a sequence $\{z(t)\}_0^\infty$ as follows:

$$z(t) = z_1\,(1, 0) \quad \text{for } t = 0, 2, 4, \ldots$$

$$z(t) = z_1\,(0, 1) \quad \text{for } t = 1, 3, 5, \ldots$$

Also, define $x(t) = Az(t)$ and $y(t + 1) = Bz(t)$ for $t \geqslant 0$. Then clearly $(x(t), y(t + 1))) \in \Omega$ for $t \geqslant 0$. Also, we can check that

$$y(t+1) - x(t+1) = z_1\,(2, 0) \quad \text{for } t = 0, 2, 4, \ldots.$$

$$y(t+1) - x(t+1) = z_1\,(3, 0) \quad \text{for } t = 1, 3, 5, \ldots.$$

Thus $\{x(t), y(t+1)\}_0^\infty$ is a programme from $\hat{x}$, and furthermore, for $T \geqslant 2$,

$$\sum_{t=1}^{T} [w(c(t)) - w(\hat{c})] \geqslant z_1$$

so $\{x, y\}_0^\infty$ is not an optimal programme from x. Thus, there is no nontrivial stationary optimal stock.

6 THE EXISTENCE OF A NONTRIVIAL STATIONARY OPTIMAL STOCK FOR A NONCONVEX TECHNOLOGY: THE AGGREGATIVE CASE

The example of section 5 is rather conclusive in destroying any hope of a general existence theorem under nonconvexities. The analysis of section 6 demonstrates that in an aggregative model with a nonconvex technology, the existence of a nontrivial stationary optimal stock can be proved with no restriction on the 'type' of nonconvexities that may be present. The framework is thus more general than the one treated in Majumdar and Mitra (1982); the method of proof is also quite different (since an entirely 'primal' approach is used) and may be of independent interest.

The framework is the same as described in section 2, except that here we have the special case of $n = 1$. For each x in $\mathcal{R}_+$, consider the set $Y(x) = \{y$ in $\mathcal{R}_+$: (x, y) is in $\Omega\}$. By Corollary 2.1, $Y(x)$ is bounded. Since Ω is closed, so is $Y(x)$. Since $(0, 0)$ is in Ω, and Ω has the 'free-disposal' property (see Assumption 2), so 0 is in $Y(x)$. Thus,

$$f(x) = \begin{array}{l} \text{Max } y \\ \text{subject to } y \text{ in } Y(x) \end{array}$$

defines a function from $\mathcal{R}_+$ to $\mathcal{R}_+$. It is generally called a *production function*. One can check that $\Omega = \{(x, y) \text{ in } \mathcal{R}_+^2\text{: } y \leqslant f(x)\}$ by using the 'free-disposal' property. It is fairly easy to verify that $f(0) = 0$, and that f is nondecreasing and upper semicontinuous on $\mathcal{R}_+$. Furthermore, there is $\beta > 0$, such that for $x > \beta$, $f(x) < x$, by Assumption 4. By Assumption 5, there is $\bar{x} > 0$, such that $f(\bar{x}) > \bar{x}$.

It was shown in section 3 that the set of golden rules is nonempty. If $(\hat{x}, \hat{y})$ is a golden rule, then $\hat{y} = f(\hat{x})$, by Assumption 7 and the definition of f. Thus, $\hat{y} - \hat{x} = f(\hat{x}) - \hat{x}$. By definition of a golden rule, $f(\hat{x}) - \hat{x} \geqslant f(x) - x$, for all $x \geqslant 0$ such that $f(x) \geqslant x$. This is because for any such x, $(x, y) \equiv (x, f(x))$ is in Ω, and $y \geqslant x$. If $x \geqslant 0$, and $f(x) < x$, then $f(\hat{x}) - \hat{x} \geqslant f(\bar{x}) - \bar{x} > 0 \geqslant f(x) - x$. Thus, in fact, if $(\hat{x}, \hat{y})$ is a golden rule, then

$$f(\hat{x}) - \hat{x} = \hat{y} - \hat{x} \geq f(x) - x \quad \text{for all } x \geq 0 \tag{10.6.1}$$

Let $A^0 = \{(x, y)\colon (x, y) \text{ is a golden rule}\}$. The set is nonempty; it is closed. It is clearly bounded. For, if $(\hat{x}, \hat{y})$ is a golden rule, then $f(\hat{x}) - \hat{x} \geq f(\bar{x}) - \bar{x} > 0$, and so $0 \leq \hat{x} \leq \beta$; by Lemma 2.1, then, $0 \leq \hat{y} \leq \beta$. Let $B^0 = \{x\colon$ there is some y, such that (x, y) is in $A^0\}$. B^0 is nonempty, closed and bounded, since A^0 is nonempty, closed and bounded. Thus a minimum element of B^0 is well-defined. For the rest of this section, call this element $\hat{x}$, and define $\hat{y} = f(\hat{x})$, $\hat{c} = f(\hat{x}) - \hat{x} = \hat{y} - \hat{x}$. Then, $(\hat{x}, \hat{y})$ is a golden rule with the *smallest* input level. (Note that $\hat{c} > 0$, and so $\hat{x} > 0$.)

For any programme $\{x(t), y(t+1)\}_0^\infty$ from $\hat{x}$, define the sequences $\{\alpha(t)\}_0^\infty$ $\{\beta(t)\}_0^\infty$ as follows

$$\alpha(t) = [\hat{x} - x(t)] \quad \text{for } t \geq 0 \tag{10.6.2}$$

$$\beta(t) = \hat{c} - [f(x(t)) - x(t)] \quad \text{for } t \geq 0 \tag{10.6.3}$$

Using (10.6.1), $\beta(t) \geq 0$ for $t \geq 0$.

We can now show that $\hat{x}$ is a nontrivial stationary optimal stock. Define a sequence $\{\hat{x}(t), \hat{y}(t+1)\}_0^\infty$ by $[\hat{x}(t), \hat{y}(t+1)] = [\hat{x}, \hat{y}]$ for $t \geq 0$. Then $\{\hat{x}(t), \hat{y}(t+1)\}_0^\infty$ is clearly a stationary programme from $\hat{x}$ with $c(t) = \hat{c}$ for $t \geq 1$. We will prove that it is also an optimal programme from $\hat{x}$.

Suppose, on the contrary, that there is a programme $\{x(t), y(t+1)\}_0^\infty$ from $\hat{x}$, a real number $\varepsilon > 0$, and an integer $\bar{N} \geq 1$, such that

$$\varepsilon \leq \sum_{t=1}^{N} [w(c(t)) - w(\hat{c})] \quad \text{for } N \geq \bar{N} \tag{10.6.4}$$

Since w is concave, so by Jensen's inequality

$$N\, w\left[\sum_{t=1}^{N} c(t)/N\right] \geq \sum_{t=1}^{N} w(c(t)) \quad \text{for } N \geq 1 \tag{10.6.5}$$

Combining (10.6.4) and (10.6.5), we have for $N \geq \bar{N}$

$$w\left[\sum_{t=1}^{N} c(t)/N\right] \geq w(\hat{c}) + (\varepsilon/N) > w(\hat{c}) \tag{10.6.6}$$

Since w is increasing, we get for $N \geq \bar{N}$

$$\left[\sum_{t=1}^{N} c(t)/N\right] > \hat{c} \tag{10.6.7}$$

Thus, for $N \geqslant \bar{N}$, we have

$$\sum_{t=1}^{N} c(t) > N\hat{c} \tag{10.6.8}$$

Observe now that for $N \geqslant 1$,

$$\sum_{t=1}^{N} c(t) = [\hat{x} - x(N)] + \sum_{t=0}^{N-1} [y(t+1) - x(t)]$$

$$\leqslant [\hat{x} - x(N)] + \sum_{t=0}^{N-1} [f(x(t)) - x(t)]$$

Thus, recalling equation (10.6.3), we have for $N \geqslant 1$,

$$\sum_{t=1}^{N} c(t) \leqq [\hat{x} - x(N)] + N\hat{c} - \sum_{t=0}^{N-1} \beta(t) \tag{10.6.9}$$

Since $\beta(t) \geqslant 0$ for $t \geqslant 0$, we have for $N \geqslant 1$,

$$\sum_{t=1}^{N} c(t) \leqslant [\hat{x} - x(N)] + N\hat{c} \tag{10.6.10}$$

Combining (10.6.8) and (10.6.10), we have for $N \geqslant \bar{N}$,

$$\alpha(N) = [\hat{x} - x(N)] > 0 \tag{10.6.11}$$

In particular, $\alpha(\bar{N}) > 0$, and so $x(\bar{N}) < \hat{x}$, while $y(\bar{N} + 1) - x(\bar{N}) \leqslant f(x(\bar{N})) - x(\bar{N}) \leqslant \hat{c}$. Thus, by the definition of $\hat{x}$, we have $f(x(\bar{N})) - x(\bar{N}) < c$, so that $\beta(\bar{N}) > 0$.

Combining (10.6.8) and (10.6.9), we have for $N \geqslant \bar{N}$,

$$\sum_{t=0}^{N-1} \beta(t) < [\hat{x} - x(N)] \tag{10.6.12}$$

Since $\beta(t) \geqslant 0$ for $t \geqslant 0$, $\beta(\bar{N}) > 0$, we have for $N \geqslant \bar{N} + 1$,

$$0 < \beta(\bar{N}) < [\hat{x} - x(N)] \tag{10.6.13}$$

Let $\tilde{x} = \hat{x} - \beta(\bar{N})$; then $x(N) \leqslant \tilde{x}$ for $N \geqslant \bar{N} + 1$. The function $g(x) \equiv [\hat{c} - [f(x) - x]]$ is lower semicontinuous (in x) on the nonempty, compact set $0 \leqslant x \leqslant \tilde{x}$. Hence g attains a minimum on this set; call this minimum value, b. Then, since $0 \leqslant x(N) \leqslant \tilde{x}$ for $N \geqslant \bar{N} + 1$, we get $\beta(N) \geqslant b$ for $N \geqslant \bar{N} + 1$. By the definition of a golden rule, $b \geqslant 0$, while by the definition of $\hat{x}$, and $\tilde{x} < \hat{x}$, it follows that $b > 0$. Using this in (10.6.12), we have for $N \geqslant \bar{N} + 1$.

$$[N - (\bar{N} + 1)]b + \beta(\bar{N}) < [\hat{x} - x(N)] \leqslant \hat{x} \qquad (10.6.14)$$

Clearly, for large N, (10.6.14) leads to a contradiction. Hence $\{\hat{x}(t), \hat{y}(t + 1)\}_0^\infty$ is an optimal programme from $\hat{x}$. Thus, $\hat{x}$ is a stationary optimal stock. Since $\hat{c} = f(\hat{x}) - \hat{x} \geqslant f(\bar{x}) - \bar{x} > 0$, and w is increasing, $w(\hat{c}) > w(0)$. Hence, $\hat{x}$ is a nontrivial stationary optimal stock.

7 THE EXISTENCE OF A PRICE-SUPPORTED GOLDEN RULE FOR A NONCONVEX TECHNOLOGY: THE MULTI-SECTORAL CASE

The main purpose of section 7 is to present a set of *sufficient* conditions on our multi-sectoral economy under which there exists a price-supported golden rule, even though the technology set is nonconvex. We also present a necessary and sufficient condition for our multi-sectoral economy to have this property. Finally, we discuss some examples which clarify the role of the above-stated conditions in our existence result.

Crucial to the development of our theory is the assumption that the technology set is 'quasi-star-shaped' with respect to some golden rule. It seems appropriate at this point to discuss this concept in some detail.

The technology set, Ω, is *star-shaped with respect to some golden rule* $(\hat{x}, \hat{y})$, if (x, y) in Ω implies that $[\lambda x + (1 - \lambda)\hat{x}, \lambda y + (1 - \lambda)\hat{y}]$ is in Ω for every $0 < \lambda < 1$. Technology sets with this property can display significant nonconvexities in production. Here is an example.

Let f: $\mathcal{R}_+ \rightarrow \mathcal{R}_+$ be defined as follows:

$$f(x) = 3x \text{ for } 0 \leqslant x \leqslant 1$$
$$f(x) = 4x - 1 \text{ for } 1 \leqslant x \leqslant \tfrac{3}{2}$$
$$f(x) = 2x + 2 \text{ for } \tfrac{3}{2} \leqslant x \leqslant 2$$
$$f(x) = (\tfrac{1}{2})\, x + 5 \text{ for } x \geqslant 2$$

The technology set, Ω, is then defined as follows:

$$\Omega = \{(x, y) \text{ in } \mathcal{R}_+^2\text{: } y \leqslant f(x)\}$$

It can be checked that Ω satisfies Assumptions 1–5. Letting w be any concave, increasing and continuous function from $\mathscr{R}_+$ to $\mathscr{R}$, w satisfies Assumption 6 and Assumption 7. It can be verified that $[f(x) - x]$ attains a maximum at $x = 2$, and that there is a unique golden rule, given by $(\hat{x}, \hat{y}) = (2, 6)$. It is easy to check formally that Ω is star-shaped with respect to the golden rule point $(\hat{x}, \hat{y})$. It is clear that Ω is a nonconvex technology set.

However, the assumption that Ω is star-shaped with respect to some golden rule excludes other interesting technologies that we would certainly like to include in the theory. Here is an example. Let f: $\mathscr{R}_+ \rightarrow \mathscr{R}_+$ be defined as follows:

$$f(x) = x + (1/2)x^2 - (1/2)x^3 \qquad \text{for } 0 \leqslant x \leqslant 1$$

$$f(x) = (1/2)x + (1/2) \qquad \text{for } x \geqslant 1$$

The technology set, Ω, is defined as follows:

$$\Omega = \{(x, y) \text{ in } \mathscr{R}^2_+: y \leqslant f(x)\}$$

Let w be any increasing, continuous and concave function from $\mathscr{R}_+$ to $\mathscr{R}$. Then (Ω, w) satisfy assumptions 1–7. It can be verified that $[f(x) - x]$ is maximised at $x = (2/3)$, and that there is a unique golden rule, defined by $(\hat{x}, \hat{y}) = [(2/3), (20/27)]$. Now, the technology set is clearly not star-shaped with respect to the golden rule, $(\hat{x}, \hat{y})$. For example, taking $(0, 0)$ in Ω, we note that the line segment joining $(0, 0)$ to $(\hat{x}, \hat{y})$ does not wholly lie in Ω. To be precise, for $0 < \lambda < (1/2)$, $[\lambda\hat{x} + (1 - \lambda) \cdot 0, \lambda\hat{y} + (1 - \lambda) \cdot 0]$ does not belong to Ω, while for $1 > \lambda \geqslant (1/2)$, $[\lambda\hat{x} + (1 - \lambda) \cdot 0, \lambda\hat{y} + (1 - \lambda) \cdot 0]$ does belong to Ω.

Now, this example represents the canonical ('S-shaped') case discussed in the aggregative model by Skiba (1978), Majumdar and Mitra (1982; 1983), Dechert and Nishimura (1983), and others. Consequently, one is reluctant to make an assumption which rules out this case. The example just discussed, however, indicates that it might be justifiable to assume that given (x, y) in Ω, a convex combination of (x, y) and $(\hat{x}, \hat{y})$, 'sufficiently close to $(\hat{x}, \hat{y})$', would also be in Ω. This leads us naturally to the notion of a 'quasi-star-shaped' technology.

The technology set, Ω, is *quasi-star-shaped with respect to some golden rule* $(\hat{x}, \hat{y})$ if (x, y) in Ω implies that there is $0 < \lambda(x, y) \leqslant 1$ such that, for $0 < \lambda < \lambda(x, y)$, $(\lambda x + (1 - \lambda)\hat{x}, \lambda y + (1 - \lambda)\hat{y})$ is in Ω. (Note that if for each (x, y) in Ω, $\lambda(x, y) = 1$, then Ω is star-shaped with respect to $(\hat{x}, \hat{y})$).

We will show, in the next section, that for the S-shaped production function case, the golden rule is unique and the technology set is quasi-star-shaped with respect to this golden rule. (In fact, a 'strict' version of this property will be verified there.) Given this, we feel somewhat justified in proceeding with the following assumption in the multi-sectoral case.

Condition QS (Quasi-Star-Shaped Technology) The technology set, Ω, is quasi-star-shaped with respect to some golden rule, $(\hat{x}, \hat{y})$.

In addition to the above condition, we also use the following two 'regularity' conditions to obtain a price-supported golden rule.

Condition PC (Positive Consumption at Golden Rule) If $(\bar{x}, \bar{y})$ is any golden rule, then $\bar{y} >> \bar{x}$.

Condition S (Smoothness of Utility Function) The utility function, w, is twice continuously differentiable on $\mathscr{R}^n_{++}$.

Proposition 7.1 Assume that Conditions QS, PC and S are satisfied. Then there is $\hat{p}$ in $\mathscr{R}^n_+$, $\hat{p} > 0$, such that

$$w(\hat{y} - \hat{x}) - \hat{p}(\hat{y} - \hat{x}) \geqslant w(c) - \hat{p}c \text{ for all } c \text{ in } \mathscr{R}^n_+ \tag{10.7.1}$$

$$\hat{p}\hat{y} - \hat{p}\hat{x} \geqslant \hat{p}y - \hat{p}x \text{ for all } (x, y) \text{ in } \Omega \tag{10.7.2}$$

Proof Note that w is concave and continuous on $\mathscr{R}^n_+$ and differentiable on $\mathscr{R}^n_{++}$, and $(\hat{y} - \hat{x}) >> 0$. So for all c in $\mathscr{R}^n_+$,

$$w(c) - w(\hat{y} - \hat{x}) \leqslant Dw(\hat{y} - \hat{x}) [c - (\hat{y} - \hat{x})]$$

Hence, defining $\hat{p} \equiv Dw(\hat{y} - \hat{x})$, we have

$$w(\hat{y} - \hat{x}) - \hat{p}(\hat{y} - \hat{x}) \geqslant w(c) - \hat{p}c \text{ for all } c \text{ in } \mathscr{R}^n_+$$

Using Assumption 7, we clearly have $\hat{p}$ in $\mathscr{R}^n_+$. Also, by Assumption 7, $\hat{p} > 0$. This establishes (10.7.1).

We will next show that for all (x, y) in Ω,

$$\hat{p}y - \hat{p}x \leqslant \hat{p}\hat{y} - \hat{p}\hat{x}$$

Suppose, on the contrary, there is some (x', y') in Ω and $\Theta > 0$, such that $[\hat{p}y' - \hat{p}x'] \geqslant [\hat{p}\hat{y} - \hat{p}\hat{x}] + \Theta$. Since Ω is quasi-star-shaped with respect to $(\hat{x}, \hat{y})$, there is $0 < \lambda\ (x', y') \leqslant 1$, such that for $0 < \lambda < \lambda(x', y')$, we have $[\lambda x' + (1 - \lambda)\hat{x}, \lambda y' + (1 - \lambda)\hat{y}]$ in Ω.

Define $B = \{c \text{ in } \mathscr{R}^n_+\colon (1/2)(\hat{y} - \hat{x}) \leqslant c \leqslant 2(\hat{y} - \hat{x})\}$. Clearly, B is a nonempty, compact subset of $\mathscr{R}^n_{++}$. Consequently, there is $b > 0$, such that for all z in B,

$$[(y' - x') - (\hat{y} - \hat{x})]D^2 w(z) [(y' - x') - (\hat{y} - \hat{x})] \geqslant -b \tag{10.7.3}$$

using the fact that w is twice-continuously differentiable on $\mathscr{R}^n_{++}$ and so, given any h in $\mathscr{R}^n$, $h'D^2w(z)h$ is continuous (in z) on B.

In view of the fact that $(\hat{y} - \hat{x}) >> 0$, one can now choose $0 < \lambda < \lambda(x', y')$ [with λ sufficiently close to 0] so that

(i) $[\lambda x' + (1 - \lambda)\hat{x}, \lambda y' + (1 - \lambda)y']$ is in Ω
(ii) $[\lambda y' + (1 - \lambda)\hat{y}] - [\lambda x' + (1 - \lambda)\hat{x}]$ is in B
(iii) $\lambda b \leqslant \Theta$

Fixing this λ, define $(x'', y'') = [\lambda x' + (1 - \lambda)\hat{x}, \lambda y' + (1 - \lambda)\hat{y}]$, $c' = (y' - x')$ in $\mathcal{R}^n$, $c'' = (y'' - x'')$ in $\mathcal{R}^n_{++}$, $\hat{c} = (\hat{y}, \hat{x})$ in $\mathcal{R}^n_{++}$, and write, by Taylor's theorem,

$$w(c'') - w(\hat{c}) = Dw(\hat{c})\,[c'' - \hat{c}] + (1/2)\,[c'' - \hat{c}]D^2\,w(z)\,[c'' - \hat{c}]$$

where z is some convex-combination of c'' and $\hat{c}$. Noting that $(c'' - \hat{c}) = \lambda(c' - \hat{c})$, we have

$$w(c'') - w(\hat{c}) = Dw(\hat{c})\,\lambda(c' - \hat{c}) + (1/2)\,\lambda^2\,(c' - \hat{c})D^2\,w(z)\,(c' - \hat{c})$$

Now, using (10.7.3) we have

$$\begin{aligned} w(c'') - w(\hat{c}) &\geqslant \lambda\Theta + (1/2)\lambda^2\,[-\,b] \\ &= \lambda[\Theta - (1/2)\lambda b] \\ &\geqslant \lambda[\Theta - (\Theta/2)] \\ &= \lambda\Theta/2 > 0 \end{aligned}$$

Hence, we have (x'', y'') in Ω with $y'' - x'' = c''$ in B (so that $y'' - x'' \geqslant 0$), and $w(y'' - x'') > w(\hat{y} - \hat{x})$. This contradicts the fact that $(\hat{x}, \hat{y})$ is a golden rule, and establishes (10.7.2).

Remark It is clear from the proof of Proposition 7.1 that we used the strict positivity of consumption only at the golden rule $(\hat{x}, \hat{y})$, with respect to which the technology set is assumed to be quasi-star-shaped in Condition QS. Thus, Condition PC can be relaxed somewhat, and in fact condition QS and PC can be replaced by the following single condition: the technology set is quasi-star-shaped with respect to some golden rule, $(\hat{x}, \hat{y})$, and $\hat{y} >> \hat{x}$. Of course, if there is a unique golden rule, there is no difference between these alternative formulations.

The above result provides a set of *sufficient conditions* under which a golden rule has a price-support. The chief advantage is that each condition can be checked fairly easily in concrete examples. At the same time, it might be useful to provide *a necessary and sufficient condition* for a golden rule to have price-support. We present below one such result which is fairly easy to obtain.

Recall that a golden rule is an input–output pair $(\hat{x}, \hat{y}) \varepsilon \Omega$ which solves the problem:

$$\left.\begin{array}{ll} \max & w(y - x) \\ \text{Subject to} & (x, y) \in \Omega \\ \text{and} & y \geqslant x \end{array}\right\} \quad (P)$$

Let con Ω denote the convex-hull of Ω, and consider the problem:

$$\left.\begin{array}{ll} \max & w(y - x) \\ \text{Subject to} & (x, y) \in \text{con } \Omega \\ \text{and} & y \geqslant x \end{array}\right\} \quad (P')$$

Proposition 7.2 Suppose $(\hat{x}, \hat{y})$ solves problem (P). Then $(\hat{x}, \hat{y})$ has a price-support if and only if $(\hat{x}, \hat{y})$ solves problem (P').

Proof First, suppose that $(\hat{x}, \hat{y})$ has a price-support $\hat{p}$. Let $(x, y) \in$ con Ω, with $y \geqslant x$. Then, there exists $(x^s, y^s) \in \Omega$ and $0 \leqslant \lambda^s \leqslant 1$, for $s = 1, 2, \ldots,$ S such that

$$\sum_{s=1}^{S} \lambda^s = 1 \text{ and } (x, y) = \sum_{s=1}^{S} \lambda^s (x^s, y^s)$$

Since $\hat{p}$ is a price-support of $(\hat{x}, \hat{y})$, we get

$$\hat{p}(\hat{y} - \hat{x}) \geqslant \hat{p}(y^s - x^s) \quad \text{for } s = 1, \ldots, S \tag{10.7.4}$$

Using the inequalities in (10.7.4), we get

$$\hat{p}(\hat{y} - \hat{x}) \geqslant \hat{p}(y - x) \tag{10.7.5}$$

Also, since $\hat{p}$ is a price-support of $(\hat{x}, \hat{y})$, we have

$$w(\hat{y} - \hat{x}) - w(y - x) \geqslant \hat{p}(\hat{y} - \hat{x}) - \hat{p}(y - x) \tag{10.7.6}$$

Using (10.7.5) and (10.7.6), $w(\hat{y} - \hat{x}) \geqslant w(y - x)$. Since $(\hat{x}, \hat{y}) \in$ con Ω and $\hat{y} \geqslant \hat{x}$, we can then conclude that $(\hat{x}, \hat{y})$ solves (P').

Next, suppose $(\hat{x}, \hat{y})$ solves (P'). Since con Ω is convex and $(\bar{x}, \bar{y}) \in$ con Ω, we can use the argument employed in Lemma 4.1 to get $\hat{p} \in \mathscr{R}^n_+$ which provides a price-support to $(\hat{x}, \hat{y})$.

In section 5, we provided an example (Example 5.1) of a multi-sectoral economy for which the (unique) golden rule does not have a price-support.

Note that in the economy described there, Condition PC is satisfied as is Condition S. But the technology set is not quasi-star-shaped with respect to the golden rule. Thus Example 5.1 provides a justification for the use of Condition QS in establishing Proposition 7.1. In terms of the characterisation provided in Proposition 7.2, we note that $(\tilde{x}, \tilde{y})$, defined by $\tilde{x} = (0.5, 0.5)$, $\tilde{y} = (3, 0.5)$, is in con Ω, with $\tilde{y} - \tilde{x} = (2.5, 0)$, and $w(\tilde{y} - \tilde{x}) = 2.5 > 2 = w(\hat{y} - \hat{x})$ so that $(\hat{x}, \hat{y})$ does not solve problem (P') (It can be checked that $(\tilde{x}, \tilde{y})$ does, in fact, solve (P').)

In the following two examples, we provide our justification for the use of our regularity conditions (Condition PC and Condition S) in establishing Proposition 7.1.

Example 7.1

$$\text{Let } A = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 1 & 2 \end{bmatrix}, B = \begin{bmatrix} 0 & 4 & 8 \\ 4 & 1 & 0 \end{bmatrix}, e = (1, 1, 1)$$

Define the technology set $\Omega = \{(x, y) \in \mathscr{R}^2_+ \times \mathscr{R}^2_+ \colon Az \leqslant x, Bz \geqslant y, ez \leqslant 1$ and $\min(z_1, z_3) = 0$ for some $z \in \mathscr{R}^3_+\}$. Define the utility function $w(c_1, c_2) = c_1 + c_2$ for $(c_1, c_2) \in \mathscr{R}^2_+$. Then (Ω, w) satisfy Assumptions 1–7.

We note, first, that $\hat{z} = (0, 1, 0)$, $\hat{x} = (1, 1)$, $\hat{y} = (4, 1)$, $\hat{c} = (3, 0)$ and $w(\hat{c}) = 3$ characterises the (unique) golden rule. To see this note that $\hat{z}_3 = 0$. For if $\hat{z}_3 > 0$, then $\hat{z}_1 = 0$, and we have $\hat{z}_2 + 2\hat{z}_3 \leqslant \hat{x}_2$ while $\hat{z}_2 \geqslant \hat{y}_2$, so that $\hat{y} \geqslant \hat{x}$ is clearly violated. Thus $\hat{z}_3 = 0$, and so $\hat{c} = \hat{y} - \hat{x} \leqslant (B - A)\hat{z} = [3\hat{z}_2 - 2\hat{z}_1, 4\hat{z}_1]$. This means $w(\hat{c}) \leqslant 3\hat{z}_2 + 2\hat{z}_1$; this last expression is clearly maximised uniquely at $\hat{z}_1 = 0$, $\hat{z}_2 = 1$. Now, it readily follows that $\hat{x} = (1, 1)$, $\hat{y} = (4, 1)$, $\hat{z} = (0, 1, 0)$, $\hat{c} = (3, 0)$ and $w(\hat{c}) = 3$ characterises the golden rule.

Second, we observe that the golden rule $(\hat{x}, \hat{y})$ does not have a price-support. To see this, suppose $\hat{p}$ were a price-support of $(\hat{x}, \hat{y})$. We define $z = (1, 0, 0)$, $x = Az = (2, 0)$; $y = Bz = (0, 4)$, so that $(x, y) \in \Omega$. Similarly we define $z' = (0, 0, 1)$, $x' = Az' = (0, 2)$ and $y' = Bz' = (8, 0)$, so that $(x', y') \in \Omega$. Using the fact that $\hat{p}\hat{y} - \hat{p}\hat{x} \geqslant \hat{p}y - \hat{p}x$, we must have $3\hat{p}_1 \geqslant 4\hat{p}_2 - 2\hat{p}_1$, so that $5\hat{p}_1 \geqslant 4\hat{p}_2$. Similarly, using the fact that $\hat{p}\hat{y} - \hat{p}\hat{x} \geqslant \hat{p}y' - \hat{p}x'$, we must also have $3\hat{p}_1 \geqslant 8\hat{p}_1 - 2\hat{p}_2$, so that $2\hat{p}_2 \geqslant 5\hat{p}_1$. Combining the two pieces of information, $\hat{p}_1 = 0$ and $\hat{p}_2 = 0$. But then, we must have

$$w(c) = w(c) - \hat{p}c \leqslant w(\hat{c}) - \hat{p}\hat{c} = w(\hat{c}), \text{ for all } c \in \mathscr{R}^2_+$$

which is contradicted by choosing, for instance, $c = (3, 1)$.

It can be checked in this example that Condition QS is satisfied as is Condition S. However, Condition PC is violated, since $\hat{c} = (3, 0)$ is the golden rule consumption vector. This example thus shows why Condition

PC was needed to establish the price-support property of the golden rule in Proposition 7.1. In terms of Proposition 7.2, we note that $(\tilde{x}, \tilde{y})$, defined by $\tilde{x} = (2/3, 4/3)$, $\tilde{y} = (16/3, 4/3)$, is in con Ω, with $\tilde{y} - \tilde{x} = (14/3, 0)$, and $w(\tilde{y} - \tilde{x}) = 14/3 > 3 = w(\hat{c})$, so that $(\hat{x}, \hat{y})$ does not solve problem (P'). (It can be checked that $(\tilde{x}, \tilde{y})$ defined above solves (P').)

Example 7.2

$$\text{Let } A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}, B = \begin{bmatrix} 3 & 4 & 5 \\ 5 & 4 & 3 \end{bmatrix}, e = (1, 1, 1)$$

Define the technology set $\Omega = \{(x, y) \in \mathscr{R}^2_+ \times \mathscr{R}^2_+ : Az \leqslant x, Bz \geqslant y, ez \leqslant 1$ and $\min\,(z_1, z_3) = 0$ for some $z \in \mathscr{R}^3_+\}$. Define the utility function $w(c_1, c_2) = \min\,[c_1, c_2]$ for $(c_1, c_2) \in \mathscr{R}^2_+\}$. Then (Ω, w) satisfy Assumptions 1–7.

We first note that there is a unique golden rule given by $\hat{x} = (1, 1)$, $\hat{y} = (4, 4)$, $\hat{z} = (0, 1, 0)$, $\hat{c} = (3, 3)$, $w(\hat{c}) = 3$. To see this, consider any $(x, y) \in \Omega$ with $y \geqslant x$ and $w(y - x) \geqslant 3$. Let z be the associated 'activity-levels' vector. Then either (i) $z_3 = 0$ or (ii) $z_1 = 0$. Consider case (i); here we have $(3 + z_1 + z_2) \leqslant (3 + x_1) \leqslant y_1 \leqslant 3z_1 + 4z_2 = 3(z_1 + z_2) + z_2 \leqslant 3 + z_2$. Thus $z_1 = 0$, and so $z = \hat{z}$, $(x, y) = (\hat{x}, \hat{y})$, $y - x = \hat{c}$ and $w(y - x) = 3$. Case (ii) can be worked out symmetrically.

Second, we note that the golden rule $(\hat{x}, \hat{y})$ does not have a price-support. Suppose $\hat{p}$ is a price support of $(\hat{x}, \hat{y})$. We can define $z = (1, 0, 0)$, $x = Az = (1, 0)$, $y = Bz = (3, 5)$. Then $(x, y) \in \Omega$ and so $\hat{p}\hat{y} - \hat{p}\hat{x} \geqslant \hat{p}y - \hat{p}x$. This yields $3\hat{p}_1 + 3\hat{p}_2 \geqslant 3\hat{p}_1 + 5\hat{p}_2 - \hat{p}_1$, so that $\hat{p}_1 \geqslant 2\hat{p}_2$. Similarly, we can define $z' = (0, 0, 1)$, $x' = Az' = (0, 1)$, $y' = Bz' = (5, 3)$. Then $(x', y') \in \Omega$ and so $\hat{p}\hat{y} - \hat{p}\hat{x} \geqslant \hat{p}y' - \hat{p}x'$. This yields $3\hat{p}_1 + 3\hat{p}_2 \geqslant 5\hat{p}_1 + 3\hat{p}_2 - \hat{p}_2$, so that $\hat{p}_2 \geqslant 2\hat{p}_1$. Combining the two pieces of information, we get $\hat{p}_2 \geqslant 2\hat{p}_1 \geqslant 4\hat{p}_2$, so that $\hat{p}_2 = 0$ and $\hat{p}_1 = 0$. But then we must have

$$w(c) = w(c) - \hat{p}c \leqslant w(\hat{c}) - \hat{p}\hat{c} = w(\hat{c}) \quad \text{for all } c \in \mathscr{R}^2_+$$

which is contradicted by choosing, for instance, $c = (4, 4)$.

It can be checked in this example that Condition QS is satisfied as is Condition PC. However, Condition S is violated, since w is not differentiable. This example thus shows why condition S was needed to establish the price-support property of the golden rule in Proposition 7.1. In terms of Proposition 7.2, we note that $(\tilde{x}, \tilde{y})$, defined by $\tilde{x} = (0.5, 0.5)$ $\tilde{y} = (4, 4)$ is in con Ω, with $\tilde{y} - \tilde{x} = (3.5, 3.5)$, and $w(\tilde{y} - \tilde{x}) = 3.5 > 3 = w(\hat{c})$, so that $(\tilde{x}, \tilde{y})$ does not solve problem (P'). (It can be checked that $(\tilde{x}, \tilde{y})$ defined above solves (P').)

8 THE EXISTENCE OF A NONTRIVIAL STATIONARY OPTIMAL STOCK FOR A NONCONVEX TECHNOLOGY: THE MULTI-SECTORAL CASE

We are now in a position to present the main existence result of this study. We establish the existence of a nontrivial stationary optimal stock in the multi-sector case, under nonconvexities in production. In obtaining this result, we will rely heavily on the price-support property of a golden rule established in Proposition 7.1. Thus, for the rest of this section, we assume that Conditions PC and S hold. For technical reasons, we actually need the technology set to satisfy a stricter condition than Condition QS. Let us say that the technology set, Ω, is *strictly quasi-star-shaped with respect to some golden rule* $(\hat{x}, \hat{y})$ if (x, y) in Ω, $(x, y) \neq (\hat{x}, \hat{y})$ implies that there is $0 < \lambda(x, y) \leqslant 1$ such that for $0 < \lambda < \lambda(x, y)$, we have $(\lambda x + (1 - \lambda)\hat{x}, y') \varepsilon \Omega$ with $y' >> \lambda y + (1 - \lambda)\hat{y}$.

We now show that for the canonical (S-shaped production function) case, the technology set satisfies this restriction in the aggregative model. Let f be a function from $\mathscr{R}_+$ to $\mathscr{R}_+$, such that $f(0) = 0$, f is increasing, twice continuously differentiable for $x \geqslant 0$, and $f'(\infty) < 1 \leqslant f'(0) < \infty$. Furthermore, suppose there is k_1 such that $0 < k_1 < \infty$; $f''(x) > 0$ for $0 \leqslant x < k_1$; $f''(x) < 0$ for $x > k_1$; and $f''(x) = 0$ for $x = k_1$. Let $\Omega = \{(x\ y) \text{ in } \mathscr{R}^2_+ : y \leqslant f(x)\}$. It can be checked that there are uniquely determined numbers $\hat{x}$, β, k_2, such that $0 < k_1 < k_2 < \hat{x} < \beta < \infty$; $f'(\hat{x}) = 1$; $f(\beta) = \beta$; and, $f'(k_2) = [f(k_2)/k_2]$. Furthermore, for $0 \leqslant x < \hat{x}$, $f'(x) > 1$; and for $x > \hat{x}$, $f'(x) < 1$; for $0 < x < \beta$, $x < f(x) < \beta$, and for $x > \beta$, $\beta < f(x) < x$; for $0 < x < k_2$, $f'(x) > [f(x)/x]$, and for $x > k_2$, $f'(x) < [f(x)/x]$. Also, for $0 < x < k_2$, $[f(x)/x]$ is increasing, and for $x > k_2$, $[f(x)/x]$ is decreasing; for $0 \leqslant x < k_1$, $f'(x)$ is increasing, and for $x > k_1$, $f'(x)$ is decreasing.

Note that $0 < k_2 < \hat{x}$, so that $0 < [k_2/\hat{x}] < 1$. Define $\mu = 1 - [k_2/\hat{x}]$; then $0 < \mu < 1$. Define $h(x) = [f(k_2)/k_2]x$ for $0 \leqslant x \leqslant k_2$; $h(x) = f(x)$ for $x \geqslant k_2$. Then h is a concave function on $\mathscr{R}_+$; furthermore it is strictly concave for $x \geqslant k_2$. Since $\hat{x} > k_2$, we have $f(\hat{x}) = h(\hat{x})$. So, $[\hat{x}, \hat{y}] \equiv [\hat{x}, f(\hat{x})]$ is a golden rule; and in fact it is the only golden rule, since h is concave, and $h''(\hat{x}) = f''(\hat{x}) < 0$.

We check now that Ω is strictly quasi-star-shaped with respect to $(\hat{x}, \hat{y})$. Let (x, y) be in Ω, with $(x, y) \neq (\hat{x}, \hat{y})$. Then defining $\lambda(x, y) = \mu$, we note that for $0 < \lambda < \lambda(x, y)$, $(1 - \lambda) > (1 - \mu) = (k_2/\hat{x})$, so that $(1 - \lambda)\hat{x} > k_2$, and so $\lambda x + (1 - \lambda)\hat{x} > k_2$.

There are now three cases to consider: (a) $x = 0$, (b) $0 < x < k_2$, (c) $x \geqslant k_2$.

In case (a), $f[\lambda x + (1 - \lambda)\hat{x}] = f[(1 - \lambda)\hat{x}] = \{f[(1 - \lambda)\hat{x}]/[(1 - \lambda)\hat{x}]\}(1 - \lambda)\hat{x} > [f(\hat{x})/\hat{x}]\ (1 - \lambda)\hat{x}$ [since $\hat{x} > (1 - \lambda)\hat{x} > k_2$, and $[f(x)/x]$ is decreasing in x for $x \geqslant k_2$] $= (1 - \lambda) f(\hat{x}) = \lambda f(x) + (1 - \lambda)f(\hat{x}) \geqslant \lambda y + (1 -$

$\lambda)\hat{y}$. Thus, $(\lambda x + (1 - \lambda)\hat{x}, y')$ is in Ω, with $y' > \lambda y + (1 - \lambda)\hat{y}$.

In case (*b*), we have $f[\lambda x + (1 - \lambda)\hat{x}] = h[\lambda x + (1 - \lambda)\hat{x}] \geqslant \lambda h(x) + (1 - \lambda)h(\hat{x}) > \lambda f(x) + (1 - \lambda) f(\hat{x})$ [using $f(x) < h(x)$ for $0 < x < k_2$] $\geqslant \lambda y + (1 - \lambda)\hat{y}$. Thus, $(\lambda x + (1 - \lambda)\hat{x}, y')$ is in Ω, with $y' > \lambda y + (1 - \lambda)\hat{y}$.

In case (c), if $x \neq \hat{x}$, $f[\lambda x + (1 - \lambda)\hat{x}] = h[\lambda x + (1 - \lambda)\hat{x}] > \lambda h(x) + (1 - \lambda)\, h(\hat{x})$ (since h is strictly concave for $x \geqslant k_2$, and $x \neq \hat{x}$) $\geqslant \lambda f(x) + (1 - \lambda)f(\hat{x}) \geqslant \lambda y + (1 - \lambda)\hat{y}$. If $x = \hat{x}$, but $y \neq \hat{y}$, then $y < f(x)$. So, $f[\lambda x + (1 - \lambda)\hat{x}] = h[\lambda x + (1 - \lambda)\hat{x}] \geqslant \lambda h(x) + (1 - \lambda)\, h(\hat{x}) \geqslant \lambda f(x) + (1 - \lambda) f(\hat{x}) > \lambda y + (1 - \lambda)\hat{y}$. Thus, in either subcase, $f[\lambda x + (1 - \lambda)\hat{x}] > \lambda y + (1 - \lambda)\hat{y}$. Thus, $(\lambda x + (1 - \lambda)\hat{x}, y')$ is in Ω, with $y' > \lambda y + (1 - \lambda)\hat{y}$. We have thus checked that in the canonical (S-shaped production function) case, Ω is indeed strictly quasi-star-shaped with respect to the golden rule for the aggregative model.

In order to obtain our main result on the existence of a nontrivial stationary optimal stock, we maintain the following assumption, for the rest of this section.

Condition SQS (Strictly Quasi-Star-Shaped Technology) The technology set, Ω, is strictly quasi-star-shaped with respect to some golden rule, $(\hat{x}, \hat{y})$.

In fact, it is easy to see that given Condition SQS, $(\hat{x}, \hat{y})$ is the only golden rule. For if (x, y) is some other golden rule $[(x, y) \neq (\hat{x}, \hat{y})]$, then there is $0 < \lambda(x, y) \leqslant 1$, such that for $0 < \lambda < \lambda(x, y)$, $[x', y'] \equiv [\lambda x + (1 - \lambda)\hat{x}, y']$ is in Ω, with $y' >> \lambda y + (1 - \lambda)\hat{y}$. But, then, $c' \equiv [y' - x'] >> \lambda[y - x] + (1 - \lambda)\,[\hat{y} - \hat{x}] \geqslant 0$. Hence, $w(c') > w[\lambda(y - x) + (1 - \lambda)\,(\hat{y} - \hat{x})]$ (by Assumption 7) $\geqslant \lambda w(y - x) + (1 - \lambda)w(\hat{y} - \hat{x}) = w(\hat{y} - \hat{x})$ (since $w(y - x) = w(\hat{y} - \hat{x})$) which contradicts the fact that $(\hat{x}, \hat{y})$ is a golden rule.

In view of this observation, there is only one golden rule, and for the rest of this section, we shall refer to it as $(\hat{x}, \hat{y})$. (Note that Condition PC now translates simply to the assumption that $\hat{y} >> \hat{x}$.)

The strategy adopted to establish our existence theorem can be roughly described as follows. Recalling the result for the case of a convex technology set (see section 4), we note that the convexity of the technology set was exploited heavily in (a) providing a 'price-support' for the golden rule; (b) proving an 'average turnpike property'. Now, our procedure is to note that convexity of Ω is not essential to (a); this result can be proved if convexity of Ω is replaced by Conditions QS, PC and S (as we showed in Proposition 7.1). Convexity of Ω does seem essential for result (b). So, we avoid this route, and rather follow the *earlier* literature (particularly Atsumi, 1965, and McKenzie, 1968) in establishing a 'value-loss' lemma (see Lemma 8.2) using the fact that Ω is *strictly* quasi-star-shaped with respect to $(\hat{x}, \hat{y})$. (This is precisely where the 'strictness' is needed.) This yields the asymptotic convergence (turnpike property) of the input–output sequence of a good programme to the golden rule input–output pair $(\hat{x}, \hat{y})$. (See Proposition 8.1.) The method of Gale (1967, Theorem 8) is very similar, although he does not explicitly prove a 'value-loss' lemma. The existence theorem can then be proved following the method used in the aggregative model by Majumdar and Mitra (1982); the method of Gale (1967, Theorem 9), when specialised to our problem, is very similar.

Given any c in $\mathcal{R}^n_+$, define $a(c) = [w(\hat{y} - \hat{x}) - \hat{p}(\hat{y} - \hat{x})] - [w(c) - \hat{p}c]$. Given any (x, y) in Ω, define $b(x, y) = [\hat{p}\hat{y} - \hat{p}\hat{x}] - [\hat{p}y - \hat{p}x]$. Clearly, $a(c) \geq 0$ for all c in $\mathcal{R}^n_+$ and $b(x, y) \geq 0$ for all (x, y) in Ω, by Proposition 7.1.

Lemma 8.1 Suppose (x, y) is in Ω. Then,

$$b(x, y) = 0 \text{ iff } (x, y) = (\hat{x}, \hat{y}) \tag{10.8.1}$$

Proof If $(x, y) = (\hat{x}, \hat{y})$, then $b(x, y) = b(\hat{x}, \hat{y}) = 0$, by definition of b. To prove the converse, suppose $(x, y) \neq (\hat{x}, \hat{y})$, but $b(x, y) = 0$. Then

$$\hat{p}\hat{y} - \hat{p}\hat{x} = \hat{p}y - \hat{p}x \tag{10.8.2}$$

Then there is $0 < \lambda(x, y) \leq 1$, such that for $0 < \lambda < \lambda(x, y)$, $(\lambda x + (1 - \lambda)\hat{x}, y')$ is in Ω, with $y' >> \lambda y + (1 - \lambda)\hat{y}$. Since $\hat{p} > 0$, we get $\hat{p}y' - \hat{p}[\lambda x + (1 - \lambda)\hat{x}] > \hat{p}[\lambda y + (1 - \lambda)\hat{y}] - \hat{p}[\lambda x + (1 - \lambda)\hat{x}] = \lambda[\hat{p}y - \hat{p}x] + (1 - \lambda)[\hat{p}\hat{y} - \hat{p}\hat{x}] = [\hat{p}\hat{y} - \hat{p}\hat{x}]$, by equation (10.8.2). This contradicts Proposition 7.1 and establishes the result.

Lemma 8.2 Let J be a nonempty, compact subset of Ω. Given any $\varepsilon > 0$, there is $\eta > 0$, such that (x, y) in J, and $\|(x, y) - (\hat{x}, \hat{y})\| \geq \varepsilon$ imply $b(x, y) \geq \eta$.

Proof If (x, y) is in J, and $\|(x, y) - (\hat{x}, \hat{y})\| \geq \varepsilon$, then $(x, y) \neq (\hat{x}, \hat{y})$, and so $b(x, y) > 0$ by Lemma 8.1. Thus, if Lemma 8.2 were not valid, there would be a sequence (x^s, y^s) in J, with $\|(x^s, y^s) - (\hat{x}, \hat{y})\| \geq \varepsilon$, such that $b(x^s, y^s) \to 0$ as $s \to \infty$. Since J is compact, there is a subsequence $(x^{s'}, y^{s'})$ of (x^s, y^s) converging to some (x^*, y^*) in J. Since $\|(x^{s'}, y^{s'}) - (\hat{x}, \hat{y})\| \geq \varepsilon$ for all s', so $\|(x^*, y^*) - (\hat{x}, \hat{y})\| \geq \varepsilon$. Hence, $b(x^*, y^*) > 0$.

On the other hand, $b(x^{s'}, y^{s'}) \to 0$ as $s' \to \infty$. Since $(x^{s'}, y^{s'})$ converges to (x^*, y^*) as $s' \to \infty$, we get $b(x^*, y^*) = 0$, a contradiction, which establishes the Lemma.

Remark Lemma 8.2 is our version of the well-known 'value-loss' lemma, introduced by Radner (1961) to study 'turnpike theorems' for 'final-state' optimal programmes. For Ramsey-optimal programmes, the technique has been applied directly by Atsumi (1965) and McKenzie (1968). McKenzie

(1968) actually establishes a generalised version, and the name 'value-loss lemma' has been popularised by him.

Proposition 8.1 If $\{x(t), y(t+1)\}_0^\infty$ is a good programme from k in $\mathscr{R}^n_+$, then

(i) $b(x(t-1), y(t)) \to 0$ as $t \to \infty$
(ii) $(x(t-1), y(t)) \to (\hat{x}, \hat{y})$ as $t \to \infty$

Proof For $t \geqslant 1$, we can write

$$\begin{aligned}[w(c(t)) - w(\hat{c})] &= \hat{p}[c(t) - \hat{c}] - a(c(t)) \\ &\leqslant \hat{p}[y(t) - x(t)] - \hat{p}[\hat{y} - \hat{x}] \\ &= \hat{p}[x(t-1) - x(t)] - b(x(t-1), y(t))\end{aligned}$$

Since $\{x(t), y(t+1)\}_0^\infty$ is good, there is Θ in $\mathscr{R}$, such that for $T \geqslant 1$,

$$\Theta \leqslant \sum_{t=1}^{T} w(c(t)) - w(\hat{c}) \leqslant \hat{p}k - \hat{p}x(T) - \sum_{t=1}^{T} b(x(t-1), y(t))$$

Thus, for $T \geqslant 1$,

$$\sum_{t=1}^{T} b(x(t-1), y(t)) \leqslant \hat{p}k - \Theta \leqslant \hat{p}k + |\Theta|$$

Since $b(x(t-1), y(t)) \geqslant 0$ for $t \geqslant 1$ (by definition of b), we have

$$\sum_{t=1}^{\infty} b(x(t-1), y(t)) \leqslant \infty$$

and $b(x(t-1), y(t)) \to 0$ as $t \to \infty$, proving (i).

Define J to be the set of (x, y) in Ω, such that $\|x\| \leqslant \max(\beta, \|k\|)$, $\|y\| \leqslant \max(\beta, \|k\|)$. By Corollary 2.3, $(x(t-1), y(t))$ is in J for $t \geqslant 1$. Clearly J is a nonempty, compact subset of Ω. If (ii) does not hold, there is $\varepsilon > 0$, and a subsequence $(x(t'-1), y(t'))$ of $(x(t-1), y(t))$, such that $\|(x(t'-1), y(t')) - (\hat{x}, \hat{y})\| \geqslant \varepsilon$ for all t'. Then, there is $\eta > 0$, such that $b(x(t'-1), y(t')) \geqslant \eta$ for all t', by Lemma 8.2. But this contradicts (i). Hence $(x(t-1), y(t)) \to (\hat{x}, \hat{y})$ as $t \to \infty$, proving (ii).

Theorem 8.1 There is a nontrivial stationary optimal stock.

Proof Define $\{\hat{x}(t), \hat{y}(t+1)\}_0^\infty$ as follows: $(\hat{x}(t), \hat{y}(t+1)) = (\hat{x}, \hat{y})$ for $t \geqslant 0$. Clearly $\{\hat{x}(t), \hat{y}(t+1)\}_0^\infty$ is a stationary programme from $\hat{x}$. We will now show that $\{\hat{x}(t), \hat{y}(t+1)\}_0^\infty$ is an optimal programme from $\hat{x}$. Suppose, on the contrary, there is a programme $\{\hat{x}(t), y(t+1)\}_0^\infty$ from $\hat{x}$, a real number $\varepsilon > 0$, and an integer $\bar{N}$, such that for $N \geqslant \bar{N}$,

$$\sum_{t=1}^{N} [w(c(t)) - w(\hat{c})] \geqslant \varepsilon \tag{10.8.3}$$

For $t \geqslant 1$, we can write, using Proposition 7.1,

$$\begin{aligned} w(c(t)) - w(\hat{c}) &\leqslant \hat{p}[c(t) - \hat{c}] \\ &= \hat{p}[y(t) - x(t)] - \hat{p}[\hat{y} - \hat{x}] \\ &\leqslant \hat{p}[x(t-1) - x(t)] \end{aligned}$$

$$\sum_{t=1}^{N} [w(c(t)) - w(\hat{c})] \leqslant \sum_{t=1}^{N} \hat{p}[x(t-1) - x(t)]$$

$$= \hat{p}[\hat{x} - x(N)] \tag{10.8.4}$$

Using Proposition 8.1, $\hat{p}[\hat{x} - x(N)] \to 0$ as $N \to \infty$, so that (10.8.4) contradicts (10.8.3). Hence $\{\hat{x}(t), \hat{y}(t+1)\}_0^\infty$ is optimal from $\hat{x}$. So $\hat{x}$ is a stationary optimal stock. It is non-trivial since $w(\hat{c}) > w(0)$, using $\hat{y} >> \hat{x}$ and Assumption 7.

Remarks

(i) Let $G = \{k \text{ in } \mathscr{R}_+^n$: there is a good programme from $k\}$. It is relatively straightforward to show, following the procedure used by Majumdar and Mitra (1982) in the aggregative model, that if k is in G, then there is an optimal programme from k. The details are, therefore, omitted. For conditions on (Ω, w) which ensure the existence of a good programme from arbitrary positive initial stocks, see Majumdar and Peleg (1990).

(ii) If $\{x(t), y(t+1)\}_0^\infty$ is an optimal programme from k in G, then it is a good programme, and so $[x(t), y(t+1)] \to (\hat{x}, \hat{y})$ as $t \to \infty$, by Proposition 8.1. This establishes the 'turnpike property' of optimal programmes.

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